Chapter 1 Vectors

Day – 1

Definitions

A physical quantity which has both magnitude and direction is defined as a vector quantity or a vector. It is generally represented by a directed line segment say $\overrightarrow{AB}$. A is called the initial point and B is called the terminal point. Length of the segment is called magnitude (also modulus) of the vector and we denote it by $|\overrightarrow{AB}|$.

Null vector

A vector having magnitude (or modulus) zero is called the null vector or the zero vector. It has all the directions.

Unit vector

A vector of unit magnitude is called a unit vector. Unit vectors are denoted by $\hat{a}, |\hat{a}| = 1$.

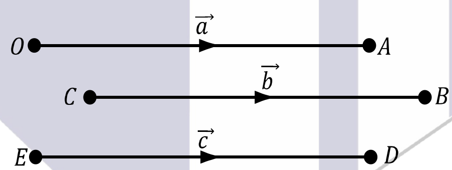
Like/Unlike vectors

Two vector of any magnitude are said to be like/unlike vector if their direction is the same/opposite respectively.

Like vectors or vectors having the same line of support are also known as collinear vectors.

Parallel vectors

Two or more vectors are said to be parallel, if they have the same support or parallel support. Parallel vectors may have equal or unequal magnitudes and direction may be same or opposite.



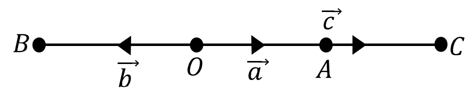
Position vector

Let O be a fixed origin, then the position vector of a point P is the vector $\overrightarrow{OP}$. If $\vec{a}$ and $\vec{b}$ are position vectors of two point A and B, then $\overrightarrow{AB} = \vec{b} - \vec{a}$ =P.V. of $B - P.V.$ of A.

Collinear vectors

Collinear vectors Here

$\overrightarrow{OA}, \overrightarrow{OB}$ *and* $\overrightarrow{OC}$ are collinear vectors



Coplanar vectors

If three or more vectors lie on the same plane these vectors are called coplanar vectors. "Two vectors are always coplanar"

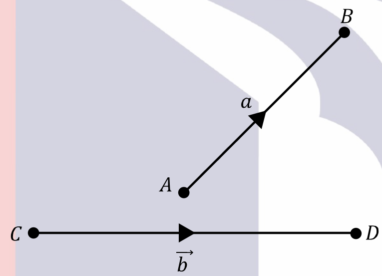
Co-initial vectors

Vectors having same initial point are called co-initial vectors.

Here, $\overrightarrow{OA}, \overrightarrow{OB}, \overrightarrow{OC}$ and $\overrightarrow{OD}$ are co-initial vectors,

Free vectors

Vectors whose initial point is not specified are called free vectors. As shown in

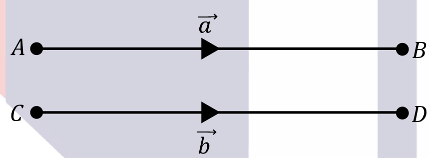


Localised vectors

A vector drawn parallel to a given vector, but through a specified point as initial point, is called localised vector

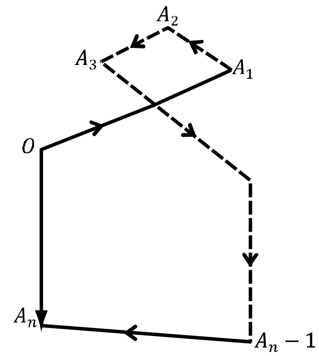
Equal vectors

Two vectors are said to be equal, if they have the same magnitude and same direction.



Polygon law of addition

$$\overrightarrow{OA}_1 + \overrightarrow{A_1A_2} + \cdots + \overrightarrow{A_{n-1}}\, A_n = \overrightarrow{OA}_n$$



Vector Addition

(i):- If two vectors $\vec{a}$ and $\vec{b}$ are represented by $\overrightarrow{OA}$ and $\overrightarrow{OB}$, then their sum $\vec{a} + \vec{b}$ is a vector represented by $\overrightarrow{OC}$ where $\overrightarrow{OC}$ is the diagonal of the parallelogram OACB.

(ii):- $\vec{a} + \vec{b} = \vec{b} + \vec{a}$ (Commutative)

(iii):- $(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c}) (Associative)$

(iv):- $\vec{a} + \vec{0} = \vec{a} = \vec{0} + \vec{a}$

(v):- $\vec{a} + (-\vec{a}) = \vec{0} = (-\vec{a}) + \vec{a}$

(vi):- $(k_1 + k_2)\,\vec{a} = k_1\,\vec{a} + k_2\,\vec{a}$ (Multiplication by Scalars)

(vii):- $k(\vec{a} + \vec{b}) = k\,\vec{a} + k\,\vec{b}$ (Multiplication by Scalars)

Multiplication Vector

In vector we define following type of products

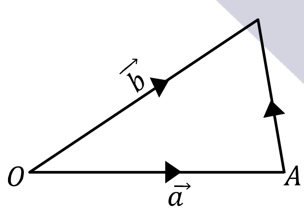
(i):- Dot product of two vectors

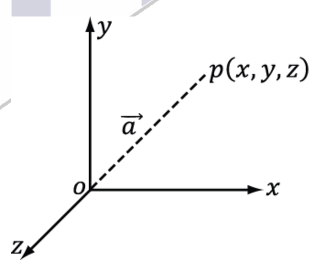
(ii):- Cross product of two vectors.

Position Vector of a Point

If a point O is fixed in a space as origin then for any point P, the vector $\overrightarrow{OP} = \vec{a}$ is called the position vector *(P.V.)* of *'P'* w.r.t *'O'*.

Let $\vec{a}\, and\, \vec{b}$ be the position vectors of the points A and B respectively with reference to the origin.

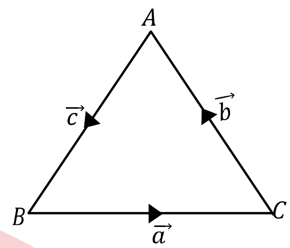




Illustration

If $\vec{a}, \vec{b}, \vec{c}$ be the vectors represented by the sides of a triangle, taken in order, then prove that $\vec{a} + \vec{b} + \vec{c} = \vec{0}$.

Solution



Let *ABC* be a triangle such that;

$$\overrightarrow{BC} = \overrightarrow{a}, \overrightarrow{CA} = \overrightarrow{b}$$
$$\overrightarrow{AB} = \vec{C}$$

Then,

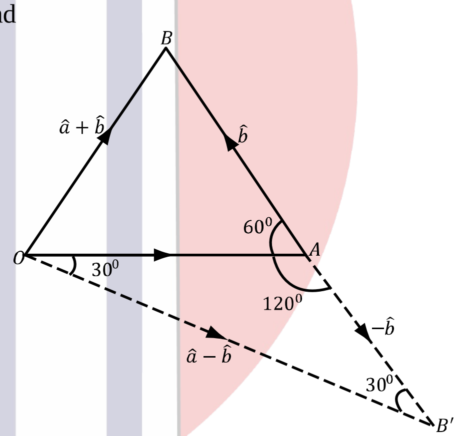
$$\vec{a} + \vec{b} + \vec{c}$$
$$\Rightarrow \quad \overrightarrow{BC} + \overrightarrow{CA} + \overrightarrow{AB}$$
$$\Rightarrow \quad \overrightarrow{BA} + \overrightarrow{AB}$$
$$\left\{\therefore \quad \overrightarrow{BC} + \overrightarrow{CA} = \overrightarrow{BA}\right\}$$
$$\vec{a} + \vec{b} + \vec{c} = 0$$

Illustration

If the sum of two unit vectors is a unit vector, prove that the magnitude of their difference is $\sqrt{3}$.

Solution

Let $\vec{a}$ *and* $\vec{b}$ two unit vectors represented by sides *OA* and *AB* of a triangle *OAB*.



Then,

$$\overrightarrow{OA} = \hat{a}, \overrightarrow{AB} = \hat{b}$$
$$\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB} = \hat{a} + \hat{b}$$

{Using triangle law of addition it is given that}

$$|\hat{a}| = |\hat{b}| = |\hat{a} + \hat{b}| = 1$$
$$\Rightarrow |\overrightarrow{OA}| = |\overrightarrow{AB}| = |\overrightarrow{OB}| = 1$$

$\Rightarrow \Delta OAB$ is equilateral triangle.

Since,

$$|\overrightarrow{OA}| = |\hat{a}| = 1 = |-\hat{b}| = |AB`|$$

Therefore**, ΔOAB` is an isosceles triangle**.

$$\Rightarrow \quad \angle AB`O = \angle AOB` = 30^0$$
$$\Rightarrow \quad \angle BOB` = \angle BOA + \angle AOB` = 60^0 + 30^0 = 90^0$$

{Since ΔBOB` is right angled}

$\therefore$ **In ΔBOB`, we have**

$$|BB`|^2 = |OB|^2 + |OB`|^2$$
$$2^2 = |\hat{a} + \hat{b}|^2 + |\hat{a} - \hat{b}|^2$$

$$2^2 = 1^2 + |\hat{a} - \hat{b}|^2$$

$$\Rightarrow \qquad |\hat{a} - \hat{b}| = \sqrt{3}.$$

Chapter 2 Three Dimensional Geometry

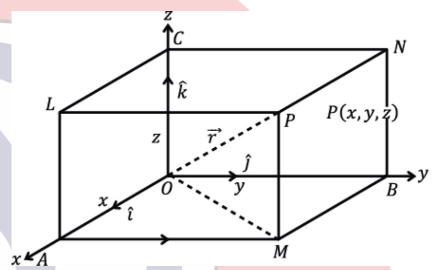
Day – 1

Three Dimensional Co-ordinate System

Position vector of a point on space

Let $\hat{i}, \hat{j}, \hat{k}$ be unit vectors (called base vectors) along OX, OY and OZ respectively.Let P(x,y,z) be a point in space, let the position of P be $\vec{r}$.

Then,



$$\vec{r} = \overrightarrow{OP}$$

$$\Rightarrow \quad \overrightarrow{OM} + \overrightarrow{MP}$$

$$\Rightarrow \quad \left(\overrightarrow{OA} + \overrightarrow{AM}\right) + \overrightarrow{MP}$$

$$\Rightarrow \quad \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}$$

$$\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$$

Thus the position vector of a point P is;

$$x\hat{\imath} + y\hat{\jmath} + z\hat{k}$$

Signs of Co-ordinates of a Point in Various Octants

Octant /Co-ordinate	*OXYZ*	*OX'YZ*	*OXY'Z*	*OXYZ'*	*OX'Y'Z*	*OX'YZ'*	*OXY'Z'*	*OX'Y'Z'*
X	+	-	+	+	-	-	+	-
Y	+	+	-	+	-	+	-	-
Z	+	+	+	-	+	-	-	-

Illustration

Planes are drawn parallel to the co-ordinate planes through the points (1,2,3) and (3,-4,-5). Find the lengths of the edges of the parallelepiped so formed.

Solution

Let P=(1,2,3) **,**Q=(3,-4,-5) through which planes are drawn parallel to the co-ordinate planes shown as,

∴PE=distance between parallel planes

ABCP and FQDE, i.e.(along z-axis)

$\Rightarrow |-5-3|$

$\Rightarrow 8$

PA= distance between parallel planes ABQF and PCDE

$\Rightarrow |3-2|$

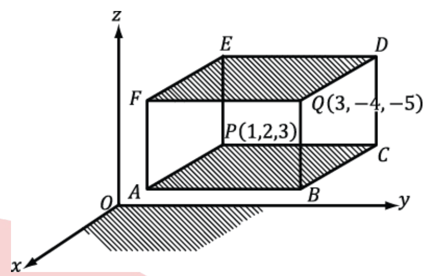
$\Rightarrow 1$

PC= distance between parallel planes BCDQ and APEF

$\Rightarrow |-4-2|$

$\Rightarrow 6$

∴lengths of edges of the parallelepiped are; 2,6,8



Distance Formula

The distance between the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is given by

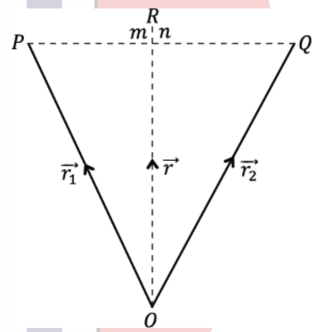
$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Section Formula

(i):- **Section formula for internal division**

If R(x,y,z) is a point dividing the join of $P(x_1,y_1,z_1)$ and $Q(x_2,y_2,z_2)$ in the ratio m:n(**Internal Div.)**
Then,

$$x = \frac{mx_2+nx_1}{m+n}, y = \frac{my_2+ny_1}{m+n}, z = \frac{mz_2+nz_1}{m+n}$$



(ii):- **Section formula for external division**

The co-ordinates of a point R which divides the join of $P(x_1,y_1,z_1)$ and $Q(x_2,y_2,z_2)$ externally in the ratio
m:n are ;

$$\left(\frac{mx_2 - nx_1}{m-n}, \frac{my_2 - ny_1}{m-n}, \frac{mz_2 - nz_1}{m-n}\right)$$

(iii):- **Mid – point formula**

The co-ordinates of the mid-point of the join of P (x_1,y_1,z_1) and $Q(x_2,y_2,z_2)$ are

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$$

Illustration

Find the ratio in which 2x+3y+5z=1 divides the line joining the points (1,0,-3) and (1,-5,7)

Solution

Here, 2x+3y+5z =1 divides(1,0,-3) and (1,-5,7) in the ratio of k:1 at point P.

Then,

$$P = \left(\frac{k+1}{k+1}, \frac{-5k}{k+1}, \frac{7k-3}{k+1}\right)$$

Which must satisfy 2x+3y+5z=1

$$\Rightarrow 2\left(\frac{k+1}{k+1}\right) + 3\left(\frac{-5k}{k+1}\right) + 5\left(\frac{7k-3}{k+1}\right) = 1$$
$$\Rightarrow 2k + 2 - 15k + 35k - 15 = k + 1$$
$$\Rightarrow 21k = 14$$
$$\Rightarrow k = 2/3$$

∴2x+3y+5z=1 divides (1,0,-3) and (1,-5,7) in the ratio of 2:3.

Illustration

Find the locus of a point the sum of whose distance from(1,0,0) and (-1,0,0) is equal to 10.

Solution

Let the points A(1,0,0), B(-1,0,0) and P(x,y,z)

Given: PA+PB=10

$$\sqrt{(x-1)^2 + (y-0)^2 + (z-0)^2} + \sqrt{(x+1)^2 + (y-o)^2 + (z-0)^2} = 10$$
$$\Rightarrow \sqrt{(x-1)^2 + y^2 + z^2}$$
$$\Rightarrow 10 - \sqrt{(x+1)^2 + y^2 + z^2}$$

Squaring both sides we get

$$\Rightarrow (x-1)^2 + y^2 + z^2$$
$$\Rightarrow 100 + (x+1)^2 + y^2 + z^2 - 20\sqrt{(x+1)^2 + y^2 + z^2}$$
$$\Rightarrow -4x - 100 = -20\sqrt{(x+1)^2 + y^2 + z^2}$$
$$\Rightarrow x + 25 = 5\sqrt{(x+1)^2 + y^2 + z^2}$$

Again squaring both sides we get

$$\Rightarrow x^2 + 50x + 625$$
$$\Rightarrow 25\{(x^2 + 2x + 1) + y^2 + z^2\}$$
$$\Rightarrow 24x^2 + 25y^2 + 25z^2 - 600 = 0$$

i.e. required equation of locus.

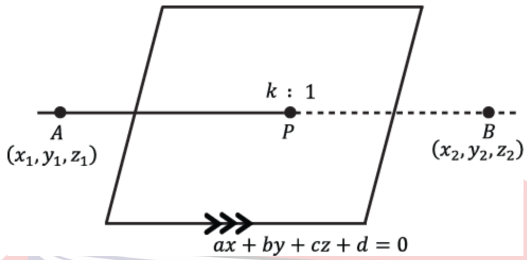
Illustration

Show that the plane ax+by+cz+d=0 divides the line joining (x_1,y_1,z_1) and (x_2,y_2,z_2) in the ratio of

$$\left(-\frac{ax_1+by_1+cz_1+d}{ax_2+by_2+cz_2+d}\right)$$

Solution

Let the plane ax+by+cz+d=0 divides the line joining (x_1,y_1,z_1) and (x_2,y_2,z_2) in the ratio of k:1 as shown in figure;



$\therefore$ co − ordinates of $P\left(\dfrac{kx_2+x_1}{k+1}, \dfrac{ky_2+y_1}{k+1}, \dfrac{kz_2+z_1}{k+1}\right)$ must satisfy $ax+by+cz+d=0$

$$a\left(\frac{kx_2+x_1}{k+1}\right)+b\left(\frac{ky_2+y_1}{k+1}\right)+c\left(\frac{kz_2+z_1}{k+1}\right)+d=0$$

$$\Rightarrow a(kx_2+x_1)+b(ky_2+y_1)+c(kz_2+z_1)+d(k+1)=0$$

$$\Rightarrow k(ax_2+by_2+cz_2+d)+(ax_1+by_1+cz_1+d)=0$$

$$k=-\frac{(ax_1+by_1+cz_1+d)}{(ax_2+by_2+cz_2+d)}.$$

<u>Direction Cosines and Direction Ratio's of a Vector</u>

1. Direction cosines

If α, β, γ are the angles which a vector $\overrightarrow{OP}$ makes with the positive directions of the co-ordinate axes OX,OY,OZ respectively, then $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are known as direction cosines of $\overrightarrow{OP}$ and are generally denoted by letters l,m,n respectively.

Thus, $l=\cos\alpha$, $m=\cos\beta$, $n=\cos\gamma$. The angles α,β,γ are known as direction angles and they satisfy the condition $0\le\alpha,\beta,\gamma\le\pi$

Direction cosines of x-axis are (1,0,0)

Direction cosines of y-axis are (0,1,0)

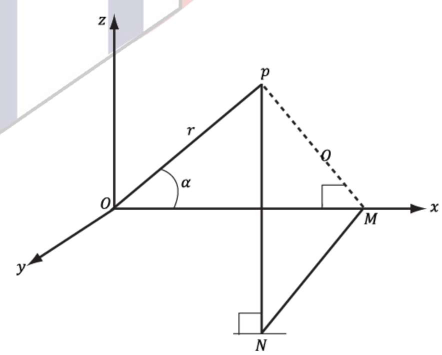
Direction cosines of z-axis are (0,0,1)

<u>Co-ordinates of P are $(r\cos\alpha, r\cos\beta, r\cos\gamma)$</u>

$$x = r\cos\alpha = lr$$

$$y = r\cos\beta = mr$$

$$z = r\cos\gamma = nr$$



(a) If l, m, n are direction cosines of a vector, then $l^2 + m^2 + n^2 = 1$

(b) $\vec{r} = |\vec{r}|\left(l\hat{\imath} + m\hat{\jmath} + n\hat{k}\right)$ and $\hat{r} = l\hat{\imath} + m\hat{\jmath} + n\hat{k}$

2. Direction Ratios

Let l, m, n be the direction cosines of a vector $\vec{r}$ and a, b, c be three numbers such that a, b, c are proportional to l, m, n

i.e,

$$\frac{l}{a} = \frac{m}{b} = \frac{n}{c}k \text{ or } (l, m, n) = (ka, kb, kc)$$

$$\Rightarrow (a, b, c) \text{are direction ratios.}$$

“That shows there can be infinitely many direction ratios for a given vector, but the direction cosines are unique”

Let a, b, c be direction ratios of a vector $\vec{r}$ having direction cosines l, m, n

Then,

$$l = \lambda a, m = \lambda b, n = \lambda c \text{ (by definition)}$$

$$= \therefore\ l^2 + m^2 + n^2 = 1$$

$$\Rightarrow a^2\lambda^2 + b^2\lambda^2 + c^2\lambda^2 = 1$$

$$\Rightarrow\ \lambda = \pm\frac{1}{\sqrt{a^2 + b^2 + c^2}}$$

So,

$$l = \pm\frac{a}{\sqrt{a^2 + b^2 + c^2}}, m = \pm\frac{b}{\sqrt{a^2 + b^2 + c^2}}, n = \pm\frac{c}{\sqrt{a^2 + b^2 + c^2}}$$

(a) If $\vec{r} = a\hat{\imath} + b\hat{\jmath} + c\hat{k}$ be a vector having direction cosines l, m, n Then

$$l = \frac{a}{|\vec{r}|}, m = \frac{b}{|\vec{r}|}, n = \frac{c}{|\vec{r}|}$$

(b) Direction ratios of the line joining two given points (x_1, y_1, z_1) and (x_2, y_2, z_2) is given by $x_2 - x_1, y_2 - y - 1\,, z_2 - z_1$.

3. Directions cosines of parallel vectors

Let $\vec{a}$ and $\vec{b}$ two parallel vectors, Then $\vec{b} = \lambda\vec{a}$ for some λ.

If $\vec{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$, *then* $\vec{b} = \lambda\vec{a}$

$$\Rightarrow \vec{b} = (\lambda a_1)\hat{\imath} + (\lambda a_2)\hat{\jmath} + (\lambda a_3)\hat{k}$$

This show that $\vec{b}$ has direction ratios $\lambda a_1, \lambda a_2, \lambda a_3$

I,e.,

$$a_1, a_2, a_3 \text{ becous } \lambda a_1 : \lambda a_2 : \lambda a_3 = a_1 : a_2 : a_3$$

Thus $\vec{a}$ and $\vec{b}$ have equal direction ratios and hence equal direction cosines also.

Straight Line In Space

Day – 2

Straight Line

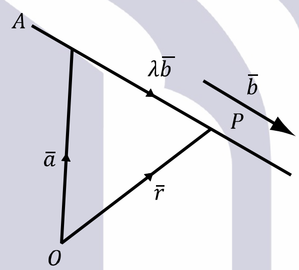
Following are the two most useful forms of the equation of a line

1. Line passing through a given point A $(\bar{a})$ and parallel to a vector $(\bar{b})$

$$r = \bar{a} + \lambda\bar{b}$$

Where $\bar{a}$ is the p.v. of any general point P on the line and λ is any real number.

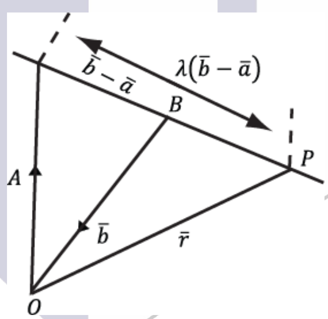
The vector equation of a straight line passing through the origin and parallel to a vector $\bar{b}$ is $\bar{r} = n\bar{b}$.



2. Line passing through two given points A $(\bar{a})$ and B $(\bar{b})$

$$\bar{r} = \bar{a} + \lambda(\bar{b} - \bar{a})$$

For each particular value of λ, we get a particular point on the line



$\bar{r} = \bar{a} + \lambda\bar{b}_1$ and $\bar{r} = \bar{a} + \mu\bar{b}_2$ the vector equation of two lines in space and θ be the angle between them, then

$$\cos\theta = \left(\frac{\bar{b}_1 . \bar{b}_2}{|\bar{b}_1||\bar{b}_2|}\right)$$

Lines are perpendicular if $b_1 . b_2 = 0$

Lines are parallel if $b_1 = \lambda\, b_2$

The internal bisector of angle between unit vectors $\hat{a}$ and $\hat{b}$ is along the vector $\hat{a} + \hat{b}$. The external bisector is along $\hat{a} - \hat{b}$.

Equation of internal and external bisectors of the lines $\bar{r} = \bar{a} + \lambda\bar{b}_1$ and $\bar{r} = \bar{a} + \mu\bar{b}_2$ (intersecting at $A(\bar{a})$ are given by

$$\bar{r} = \bar{a} + t\left(\frac{\bar{b}_1}{|\bar{b}_1|} \pm \frac{\bar{b}_2}{|\bar{b}_2|}\right).$$

Cartesian equation of straight line

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$$

Equation of straight line farms trough (x,y,z) and parallel to vector ranges DR<a,b,c>

Equation of x-axis

$$\frac{x-0}{1} = \frac{y-0}{0} = \frac{z-0}{0}$$

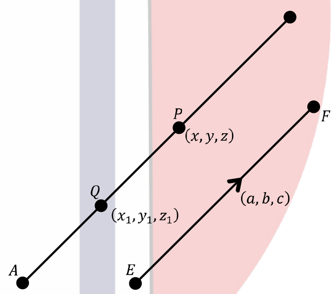
Or y =0 and z =0

Equation of y-axis

$$\frac{x-0}{0} = \frac{y-0}{1} = \frac{z-0}{1}$$

Or

$$x = 0 \text{ and } z = 0$$



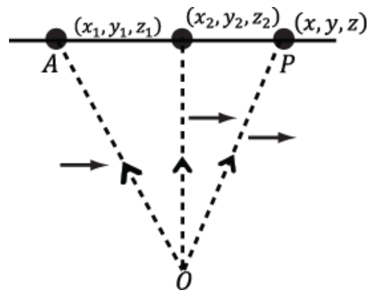
Equation of z-axis

$$\frac{x-0}{0} = \frac{y-0}{0} = \frac{z-0}{1}$$

Cartesian form

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$$

Equation Straight line farms through (x_1,y_1,z_1) and (x_2,y_2,z_2)



Illustration

Find the Cartesian equation of line are 6x-2= 3y+1=2z-2. Find its direction ratios and also find vector equation of the line.

Solution

We know

$$\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$$

is cartesian equation of straight line

$$\therefore 6x - 2 = 3y + 1 = 2z - 2$$

$$\Rightarrow 6\left(x - \frac{1}{3}\right) = 3\left(y + \frac{1}{3}\right) = 2(z-1)$$

$$\Rightarrow \frac{x-1/3}{1/6} = \frac{y+\frac{1}{3}}{1/3} = \frac{z-1}{1/2}$$

Or

$$\frac{x-1/3}{1} = \frac{y+1/3}{2} = \frac{z-1}{3}$$

Which shows given line passes through $(1/3, -1/3, 1)$ and has direction ratios (1, 2, 3) in vector form

$$\vec{a} = \frac{1}{3}\hat{i} - \frac{1}{3}\hat{j} + \hat{k} \text{ and } \vec{b} = \hat{i} + 2\hat{j} + 3\hat{k}$$

$\therefore$ its vector equation is

$$\vec{r} = \left(\frac{1}{3}\hat{i} - \frac{1}{3}\hat{j} + \hat{k}\right) + \lambda\left(\hat{i} + 2\hat{j} + 3\hat{k}\right)$$

Illustration

Prove that the line $x = ay + b, z = cy + d$ and $x = a'y + b', z = c'y + d'$ are perpendicular if $aa' + cc' + 1 = 0$

Solution

We can write the equations of straight line as

$$\frac{x-b'}{a'} = y, y = \frac{z-d'}{c'}$$

$$\Rightarrow \frac{x-b'}{a'} = \frac{y-0}{1} = \frac{z-d'}{c'}$$

And

$$\frac{x-b}{a} = y, y = \frac{z-d}{c}$$

$$\Rightarrow \frac{x-b}{a} = \frac{y-0}{1} = \frac{z-d}{c}$$

We know

$$\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$$

And

$$\frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2}$$

Are perpendicular if

$$a_1a_2 + b_1b_2 + c_1c_2 = 0$$

$\therefore$ (i) reduces to

$$aa' + i + cc' = 0$$

(as the lines are perpendicular;given).

Illustration

A line passes through $(2,-1,3)$ and is perpendicular to the line; $\vec{r} = (\hat{i}+\hat{j}-\hat{k}) + \lambda(2\hat{i}-2\hat{j}+\hat{k})$ and $\vec{r} = (2\hat{i}-\hat{j}+3\hat{k}) + \mu(\hat{i}+2\hat{j}+2\hat{k})$ obtain its equation.

Solution

To find a straight line perpendicular to given lines, $\vec{r} = \vec{a_1} + \lambda\,\vec{b_1}$ and $\vec{r} = \vec{a_2} + \lambda\,\vec{b_2}$ has dr's proportional to $\vec{b} = \vec{b_1} \times \vec{b_2}$

Now

$$\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -2 & 1 \\ 1 & 2 & 2 \end{vmatrix} = -6\,\hat{i} - 3\hat{j} + 6\hat{k}$$

Thus, the required line passes through the point (2,-1,3) and is parallel to the vector, $\vec{b} = -6\hat{i} - 3\hat{j} + 6\hat{k}$

$\therefore$ its equation is

$$\vec{r} = \frac{(2\hat{i}-\hat{j}+3\hat{k})}{(2\hat{i}-\hat{j}+3\hat{k})} + \lambda(-6\hat{i} - 3\hat{j} + 6\hat{k})$$

1. <u>Perpendicular distance of a point from a line</u>

(a) <u>Cartesian form</u>

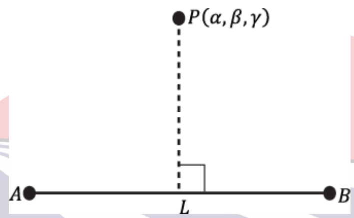
Here,

$$\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}$$

Let 'L' be the foot of perpendicular drawn from $P(\alpha,\beta,\gamma)$ on the line,

$(x-x_1)/a=(y-y_1)/b=(z-z_1)/c$ let the co-ordinates of L be

$$(x_1+a\lambda-\alpha, y_1+b\lambda-\beta, z_1+c\lambda-\gamma)$$



Also direction ratios of AB are (a,b,c)

Since PL is perpendicular to AB

$\Rightarrow a(x_1+a\lambda-\alpha)+b(y_1+b\lambda-\beta)+c(z_1+c\lambda-\gamma)=0$

$\Rightarrow \lambda=\frac{a(\alpha-x_1)+b(\beta-y_1)+c(\gamma-z_1)}{a^2+b^2+c^2}$

Putting the value of λ in $(x_1+a\lambda, y_1+b\lambda, z_1+c\lambda)$, we would get foot of perpendicular.

(b) <u>Vector form</u>

Let L be the foot of perpendicular drawn from $P(\vec{\alpha})$ on the line,

$$\vec{r}=\vec{a}+\lambda\,\vec{b}$$

Since, $\vec{r}$ denotes the position vector of any point on the line

$$\vec{r}=\vec{a}+\lambda b$$

So the position vector of L be $\left(\vec{a}+\lambda\,\vec{b}\right)$

$$\therefore \text{dr}'\text{sof PL}=\vec{a}-\vec{\alpha}+\lambda\,\vec{b}$$

Since $\overrightarrow{PL}$ is perpendicular to $\vec{b}$

$$\Rightarrow \overrightarrow{PL}\perp\vec{b}$$

$$\Rightarrow \left(\vec{a}-\vec{\alpha}+\lambda\,\vec{b}\right).\vec{b}=0$$

$$\Rightarrow (\vec{a}-\vec{\alpha}).\vec{b}+\lambda\,\vec{b}.\vec{b}=0$$

$$\Rightarrow \lambda=\frac{-(\vec{a}-\vec{\alpha}).\vec{b}}{|\vec{b}|^2}$$

Chapter 4 Plane

Day-1

Plane

A plane is a surface such that if any two points on it are taken on it, the line segment joining them lies completely on the surface.

$ax + by + cz + d = 0$ is the general equation of a plane

Equation of a plane passing through a given point

$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$, where a, b, c are constants.

Intercept form of a plane

The equation of a plane intercepting lengths a, b and c with x-axis, y-axis and z-axis respectively is

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

Illustration

A plane meets the co-ordinate axis in A, B, C such that the centroid of the ΔABC is the point (p, q, r) show that the equation of the plane is

$$\frac{x}{p} + \frac{y}{q} + \frac{z}{r} = 3$$

Solution

Let the required equation of plane be

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \qquad \ldots(i)$$

Then, the co-ordinates of A, B and C are $A(a, 0, 0)$ $B(0, b, 0)$ $C(0, 0, c)$ respectively

So the centroid of the triangle ABC

$$\left(\frac{a}{3}, \frac{b}{3}, \frac{c}{3}\right)$$

But the co-ordinate of the centroid are (p, q, r)

$$\frac{a}{3} = p, \frac{b}{3} = q, \frac{c}{3} = r$$

Putting the values of a, b and c in (i), we get

The required plane as

$$\frac{x}{3p} + \frac{y}{3q} + \frac{z}{3r} = 1$$

$$\frac{x}{p}+\frac{y}{q}+\frac{z}{r}=3$$

(i) Vector equation of a plane passing through a given point and normal to a given vector

$$(\vec{r}-\vec{a}).\vec{n}=0$$
$$\vec{r}.\vec{n}=d$$

Cartesian form

$$a(x-x_1)+b(y-y_1)+c(z-z_1)=0$$

(ii) Equation of plane in normal form vector form

The vector equation of a plane normal to unit vector $\hat{n}$ and at a distance d from the origin is $\vec{r}.\hat{n}=d$

Cartesian form

If l, m, n be the direction cosines of the normal to a given plane and p be the length of perpendicular from origin to the plane, then the equation of the plane is $lx+my+nz=p$

Illustration

Find the vector equation of plane which is at a distance of 8 units from the origin and which is normal to the vector $2\hat{i}+\hat{j}+2\hat{k}$

Solution

Here,

$$d=8 \text{ and } \vec{n}=2\hat{\imath}+\hat{\jmath}+2\hat{k}$$

$$\hat{n}=\frac{\vec{n}}{|\vec{n}|}=\frac{2\hat{i}+\hat{j}+2\hat{k}}{\sqrt{2^2+1^2+2^2}}=\frac{2\hat{i}+\hat{j}+2\hat{k}}{3}$$

Hence, the required equation of plane is

$$\vec{r}.\hat{n}=d$$

$$\Rightarrow \vec{r}.\left(\frac{2\hat{i}+\hat{j}+2\hat{k}}{3}\right)=8$$

Or

$$\vec{r}.(2\hat{i}+\hat{j}+2\hat{k})=24$$

Illustration

Reduce the equation $\vec{r}.(3\hat{i}-4\hat{j}+12\hat{k})=5$ to normal form and hence find the length of perpendicular from the origin to the plane.

Solution

The given equation of plane is

$$\vec{r}.\left(3\hat{i}-4\hat{j}+12\hat{k}\right)=5$$

Or

$$\vec{r}.\vec{n}=5$$

Where,

$$\vec{n}=3\hat{i}-4\hat{j}+12\hat{k}$$

Since, $|\vec{n}|=\sqrt{9+16+144}=13\neq 1$, therefore the given equation is not the normal form. To reduce to normal form we divide both sides by $|\vec{n}|$ i.e.,

$$\frac{\vec{r}.\vec{n}}{|\vec{n}|}=\frac{5}{|\vec{n}|}\text{ or }\vec{r}.\left(\frac{3}{13}\hat{i}-\frac{4}{13}\hat{j}+\frac{12}{13}\hat{k}\right)=\frac{5}{13}.$$

This is the normal form of the equation of given plane and length of perpendicular $=5/13$

Angle between the two planes

$$\theta=\frac{\vec{n_1}.\vec{n_2}}{|\vec{n_1}||\vec{n_2}|}$$

$$\vec{n_1}.\vec{n_2}=0$$

$$\frac{\vec{n_1}}{\vec{n_2}}=\lambda$$

$$\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}=\lambda$$

Angle between a line and a plane

$$\theta=\frac{a\alpha+b\beta+c\gamma}{\sqrt{a^2+b^2+c^2}\sqrt{\alpha^2+\beta^2+\gamma^2}}$$

$$\theta=\frac{\vec{b}.\vec{n}}{|\vec{b}||\vec{n}|}$$

Equation of Plane forming through three given points

Vector form of equation of the plane passing through three points A, B, C having position vector $\vec{a},\vec{b},\vec{c}$ respectively.

Let $\vec{r}$ be the position vector of any point P in the plane

Hence the vectors

$$\overrightarrow{AP}=\vec{r}-\vec{a},\overrightarrow{AB}=\vec{b}-\vec{a},\overrightarrow{AC}=\vec{c}-\vec{a}\ ,$$

Hence,

$$(\vec{r}-\vec{a}).\{(\vec{b}-\vec{a})\times(\vec{c}-\vec{a})\}=0$$

$$\Rightarrow (\vec{r}-\vec{a}).(\vec{b}\times\vec{c}-\vec{b}\times\vec{a}-\vec{a}\times\vec{c}+\vec{a}\times\vec{a})$$

$$\Rightarrow (\vec{r}-\vec{a}).(\vec{b}\times\vec{c}+\vec{a}\times\vec{b}+\vec{c}\times\vec{a})=0$$

$$\Rightarrow \vec{r}.(\vec{b}\times\vec{c}+\vec{a}\times\vec{b}+\vec{c}\times\vec{a})$$

$$=\vec{a}.(\vec{b}\times\vec{c})+\vec{a}.(\vec{a}\times\vec{b})+\vec{a}.(\vec{c}\times\vec{a})$$

$$\Rightarrow [\vec{r}\ \vec{b}\ \vec{c}]+[\vec{r}\ \vec{a}\ \vec{b}]+[\vec{r}\ \vec{c}\ \vec{a}]=[\vec{a}\ \vec{b}\ \vec{c}]$$

Which is required equation of plane

Cartesian equivalence

Let

$$\vec{a}=(x_1,y_1,z_1)\ \vec{b}=(x_2,y_2,z_2)\ \vec{c}=(x_3,y_3,z_3)\text{and}$$

$$\vec{r}=(x,y,z)$$

The equation of plane through three points is

$$[\vec{r}\ \vec{a}\ \vec{b}]+[\vec{r}\ \vec{b}\ \vec{c}]+[\vec{d}\ \vec{c}\ \vec{a}]=[\vec{a}\ \vec{b}\ \vec{c}]$$

$$\begin{vmatrix} x & y & z \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}+\begin{vmatrix} x & y & z \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}+\begin{vmatrix} x & y & z \\ x_3 & y_3 & z_3 \\ x_1 & y_1 & z_1 \end{vmatrix}$$

$$\Rightarrow \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$$

Equation of plane that passes through a point A with position vector $\vec{\ }$ and is parallel to given vector $\vec{\ }$ and $\vec{\ }$

Let $\vec{r}$ be the position vector of any point P in the plane

Then,

$$\overrightarrow{AP}=\overrightarrow{OP}-\overrightarrow{OA}=\vec{r}-\vec{a}$$

Since, the vectors $\vec{r}-\vec{a},\vec{b},\vec{c}$ are coplanar.

Thus,

$$(\vec{r}-\vec{a}).(\vec{b}\times\vec{c})=0$$

$$\vec{r}.(\vec{b}\times\vec{c})=\vec{a}.(\vec{b}\times\vec{c})$$

$$[\vec{r}\ \vec{b}\ \vec{c}]=[\vec{a}\ \vec{b}\ \vec{c}]$$

Which is required equation of plane.

Cartesian form

Equation of the plane passing through a point (x_1, y_1, z_1) and parallel to two lines having direction ratios $(\alpha_1, \beta_1, \gamma_1)$ and $(\alpha_2, \beta_2, \gamma_2)$ is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \end{vmatrix} = 0$$

Illustration

Find equation of plane passing through the points $P(1,1,1)$ $Q(3,-1,-2)$ $R(-3,5,-4)$

Solution

Let the equation of plane passing through $(1,1,1)$ be

$a(x-1) + b(y-1) + c(z-1) = 0$, as it passes through the point Q and R

$$\therefore 2a - 2b + c = 0$$

$$-4a + 4b - 5c = 0$$

Hence, solving by cross multiplication method, we get

$$\frac{a}{10-4} = \frac{b}{-4+10} = \frac{c}{8-8} = k$$

$$\therefore a = 6k, b = 6k, c = 0$$

Substituting in (i), we get

$$6(x-1) + 6(y-1) + 0 = 0$$

I.e., $x + y = 2$, which is the required equation.

Aliter

Equation of plane passing through $(x_1, y_1, z_1)(x_2, y_2, z_2)$ and (x_3, y_3, z_3)

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x-x_2 & y-y_2 & z-z_2 \\ x-x_3 & y-y_3 & z-z_3 \end{vmatrix} = 0$$

i.e.,

$$\begin{vmatrix} x-1 & y-1 & z-1 \\ x-3 & y+1 & z-2 \\ x+3 & y-5 & z+4 \end{vmatrix} = 0$$

On solving we get

$$x + y = 2$$

Equation of any Plane Passing through the Line of Intersection of Plane + + + = and + + + = is (+ + +) + (+ + + −) =

Chapter 5

Sphere

Day – 1

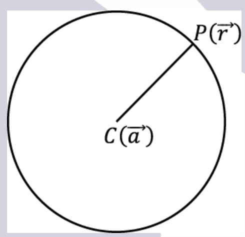
Definition

A sphere is the locus of a point which moves in space in such a way that its distance from a fixed point always remains constant.

Equation of a Sphere

Vector equation of a Sphere

The vector equation of a sphere of radius R and centre having position vector $\vec{a}$, is $|\vec{r}-\vec{a}| = R$



Remark-1

If the centre of the sphere is the origin and radius R, then its equation is

$$|\vec{r}| = R$$

Remark-2

The equation $|\vec{r}-\vec{a}| = R$ can also be written as

$$|\vec{r}-\vec{a}|^2 = R^2 \text{ or } (\vec{r}-\vec{a}).(\vec{r}-\vec{a}) = R^2$$

Illustration

Find the vector equation of a sphere with centre having the position vector $\hat{i}+\hat{j}+\hat{k}$ and radius $\sqrt{3}$

Solution

We know that the vector equation of a sphere with centre having the position vector $\vec{a}$ and radius R is

$$|\vec{r}-\vec{a}| = R$$

Here, $\vec{a} = \hat{i}+\hat{j}+\hat{k}$ and $R = \sqrt{3}$. So, the equation of the required sphere is

$$\left|\vec{r}-\left(\hat{\imath}+\hat{\jmath}+\hat{k}\right)\right|=\sqrt{3}$$

Illustration

Find the vector equation of a sphere concentric with the sphere $\left|\vec{r}+\left(\hat{\imath}-2\hat{\jmath}-3\hat{k}\right)\right|=5$ and of double its radius.

Solution

The equation of the given sphere is

$$\left|\vec{r}+\left(\hat{\imath}-2\hat{\jmath}-3\hat{k}\right)\right|=5 \text{ or } \left|\vec{r}-\left(-\hat{\imath}+2\hat{\jmath}+3\hat{k}\right)\right|=5 \ldots (i)$$

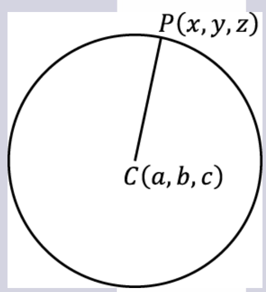
The position vector of the centre C of sphere (i) is $-\hat{\imath}+2\hat{\jmath}+3\hat{k}$ and radius = 5. Since required sphere is concentric with (i) and is of double its radius, therefore position vector of the centre of the required sphere is $-\hat{\imath}+2\hat{\jmath}+3\hat{k}$ and radius = 10. Hence, its equation is

$$\left|\vec{r}-\left(-\hat{\imath}+2\hat{\jmath}+3\hat{k}\right)\right|=10 \Rightarrow \left|\vec{r}+\left(\hat{\imath}-2\hat{\jmath}-3\hat{k}\right)\right|=10$$

Cartesian equation of a Sphere

The equation of a sphere with centre (a, b, c) and radius R is

$$(x-a)^2+(y-b)^2+(z-c)^2=R^2 \ldots (i)$$



Remark-1

The above equation is called the central form of a sphere. If the centre is at the origin, then equation (i) takes the form

$$x^2+y^2+z^2=R^2$$

Which is known as the standard form of the equation of the sphere?

Remark-2

Equation (i) Can be written

$$x^2+y^2+z^2-2ax-2by-2cz+(a^2+b^2+c^2+R^2)=0$$

Form this equation, we note the following characteristics of the equation of a sphere

(i) It is a second degree equation in x, y, z

(ii)The coefficients of x^2, y^2, z^2, are all equal
(iii)The terms containing the products xy, yz, and zx are absent

Illustration

Find the equation of a sphere whose centre is (1, 2, 3) and radius 4,

Solution

The required equation of the sphere is

$$(x-1)^2+(y-2)^2+(z-3)^2=4^2$$
$$\Rightarrow x^2+y^2+z^2-2x-4y-6z-2=0$$

Illustration

Find the equation of the sphere whose centre is C (5, -2, 3) and which passes through the point P(8, -6 ,3)

Solution

We have radius $= CP=\sqrt{(8-5)^2+(-6+2)^2+(3-3)^2}=5.$
Therefore, equation of the required sphere is

$$(x-5)^2+\left(y-(-2)\right)^2+(z-3)^2=5^2$$
$$\Rightarrow \mathrm{x}^2+\mathrm{y}^2+\mathrm{z}^2-10\mathrm{x}+4\mathrm{y}-6\mathrm{z}+13=0$$

General Equation of a Sphere

The equation $x^2+y^2+z^2+2ux+2vy+2wz+d=0$ represents a sphere with centre $(-u-v,-w)$ $i.e.$ $\left(-(1/2)\ coff.of\ x,-(1/2)coeff.of\ y,-(1/2)coeff.ofz\right)$

And, radius $=\sqrt{{}^2+{}^2+{}^2-}$

$$=\sqrt{\left(\tfrac{1}{2}.\right)^2+\left(\tfrac{1}{2}.\right)^2+\left(\tfrac{1}{2}.\right)^2-}$$

Remark

The equation $x^2+y^2=z^2+2ux+2vy+2wz+d=0$ represents a real sphere if $u^2+v^2+w^2-d>0. if\ u^2+v^2+w^2-d=0$, then it represents a point sphere.
The sphere is imaginary if $u^2+v^2+w^2-d<0$.

Illustration

Find the centre and radius of the sphere $2\mathrm{x}^2+2\mathrm{y}^2+2\mathrm{z}^2-2\mathrm{x}+4\mathrm{y}+2\mathrm{z}+3=0$

Solution

The given equation is $2x^2+2y^2+2z^2-2x+4y+2z+3=0$
Dividing throughout by 2, we get

$$x^2 + y^2 + z^2 - x + 2y + z + \frac{3}{2} = 0 \ldots (i)$$

The coordinates of the centre of (i) are

$$\text{Radius} = \sqrt{\left(\frac{1}{2}\text{coeff. of}x\right)^2 + \left(\frac{1}{2}\text{coeff. of}y\right)^2 + \left(\frac{1}{2}\text{coeff. of}z\right)^2 - \text{constant term}}$$

$$= \sqrt{\left(-\frac{1}{2}\right)^2 + (1)^2 + \left(\frac{1}{2}\right)^2 - \frac{3}{2}} = \sqrt{\frac{1}{4} + 1 + \frac{1}{4} - \frac{3}{2} = 0}$$

Thus, the given sphere represents a point sphere.

Illustration

Find the equation of the sphere concentric with $x^2 + y^2 + z^2 - 2x - 4y - 6z - 11 = 0$ but of double the radius coordinates of the centre of the required circle are also (1, 2, 3)

The radius of the given circle is $\sqrt{(-1)^2 + (-2)^2 + (-3)^2 - (-11)} = 5$. Therefore radius of the required circle is 10 units. Hence, the equation of the required circle is

$$(x-1)^2 + (y-2)^2 + (z-3)^2 = 10^2$$

$$\Rightarrow \mathrm{x}^2 + \mathrm{y}^2 + \mathrm{z}^2 - 2\mathrm{x} - 4\mathrm{y} - 6\mathrm{z} - 86 = 0$$

Illustration

Find the coordinates of the centre and the radius of the sphere whose vector equation is

$$\vec{r}^2 - \vec{r}.\left(8\hat{\imath} - 6\hat{\jmath} + 10\hat{k}\right) - 50 = 0$$

Solution

Since $\vec{r}$ denotes the position vector of any point on the sphere, therefore $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$.
The given equation is

$$\vec{r}^2 - \vec{r}.\left(8\hat{\imath} - 6\hat{\jmath} + 10\hat{k}\right) - 50 = 0$$

$$\Rightarrow \vec{r}.\vec{r} - \vec{r}.\left(8\hat{\imath} - 6\hat{\jmath} + 10\hat{k}\right) - 50 = 0$$

$$\Rightarrow \left(x\hat{\imath} + y\hat{\jmath} + z\hat{k}\right).\left(x\hat{\imath} + y\hat{\jmath} + z\hat{k}\right)$$

$$-\left(x\hat{\imath} + y\hat{\jmath} + z\hat{k}\right).\left(8\hat{\imath} - 6\hat{\jmath} + 10\hat{k} - 50 = 0\right)$$

$$\Rightarrow x^2 + y^2 + z^2 - 8x + 6y - 10z - 50 = 0$$

$\therefore$ Coordinates of the centre are (4, -3, 5) and radius $= \sqrt{16 + 9 + 25 + 50} = 10$

Self Efforts

1. Find the vector equation of the sphere concentric with the sphere $\left|\vec{r} - (\hat{i} + 2\hat{j} - 3\hat{k})\right| = 2$ and of double its radius.

2. If the position vector of one end A of a diameter AB of the sphere $\left|\vec{r} - (2\hat{i} - \hat{j} + 4\hat{k})\right| = 5$ is $\hat{i} + 2\hat{j} - \hat{k}$. Find the position vector of B.

3. Find the centre and radius of each of the following spheres

(i) $x^2 + y^2 + z^2 + 4x - 8y + 6z + 4 = 0$ [***CBSE* 1995**]

(ii) $2x^2 + 2y^2 + 2z^2 - 2x + 6y + 2z + 3 = 0$ [***CBSE* 1990**]

(iii) $x^2 + y^2 + z^2 - 4x + 6y - 8z + 29 = 0$ [***HPSB* 1994**]

4. Find the equation of the sphere concentric with the sphere $3x^2 + 3y^2 + 3z^2 - 6x + 9y - 3z + 11 = 0$ but having radius 5.

5. Find the centre and radius of the sphere whose vector equation is $\vec{}^{\,2} - \vec{}.(4\hat{} + 2\hat{} - 6\hat{}) - 11 = 0$

Answers

1. $\left|r - (\hat{\imath} + 2\hat{\jmath} - 3\hat{k})\right| = 4$ **2.** $3\hat{\imath} - 4\hat{\jmath} + 9\hat{k}$ **3.** $(i)(-2,4,-3), 5$

$(ii)\ (-2,4,-1), \sqrt{26}$ $(iii)\ (2,-3,4), 0$ **4.** $2(x^2 + y^2 + z^2) - 4x + 6y - 2z - 43 = 0$